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§6.2: Solution of initial value problems.

Thm 6.11: Suppose that $f: [0,+\infty) \to \mathbb{R}$ is continuous and the derivative f' is piecewise continuous and f is of exponential order a. Then the Laplace transform of f' exist for $s > a$ and $L(f')(s) = sL(f)(s) - f(0)$.

pf: Since f is of exponential order, by Thm 6.8, the Laplace transform of f exists for $s > a$. Fix $R > 0$, by integrating by parts, we have that $\int_0^R e^{-st} f'(t)dt = e^{-st}f(t)\Big|_0^R + s\int_0^R e^{-st} f(t)\,dt = e^{-Rs}f(R) - f(0) + s\int_0^R e^{-st} f(t)dt$. Now, $\lim_{R\to\infty} e^{-Rs} f(R) \le \lim_{R\to\infty} e^{-Rs}|f(R)| \le \lim_{R\to\infty} e^{-Rs} e^{aR} = 0$, for every $s > a$

$\Rightarrow \lim_{R\to\infty} \int_0^R e^{-st} f'(t)dt$ exists and $\lim_{R\to\infty}\int_0^R e^{-st} f'(t)dt = sL(f)(s) - f(0)$. The theorem follows.

Corollary 6.12: Suppose that $f: [0,+\infty) \to \mathbb{R}$ is a function such that $f, f', f'', \ldots, f^{(n-1)}$ are continuous of exponential order

a; and $f^{(n)}$ is piecewise continuous. Then, $L(f^{(n)})(s)$ exists for all $s > a$ and $L(f^{(n)})(s) = s^n L(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s f^{(n-2)}(0) -$ ▨

pf: We only prove $n=2$. For general n, the proof is similar.

By Thm 6.11, $L(f^{(2)})(s) = s L(f')(s) - f'(0)$. By Thm 6.11 again $L(f')(s) = sL(f)(s) - f(0)$ $\Rightarrow L(f^{(2)})(s) = s^2 L(f)(s) - sf(0) - f'(0)$ ▨

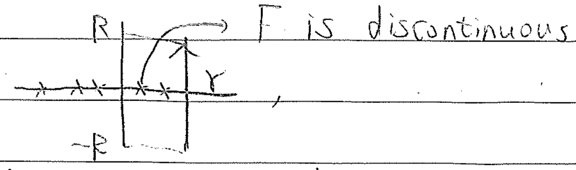
Example 6.13: Consider the O.D.E. $y'' - y' - 2y = 0$. If the solution y and its derivative y' are of exponential order a for some $a \in \mathbb{R}$, then by taking the Laplace transform of $y'' - y' - 2y = 0$, we get $[s^2 L(y) - sy(0) - y'(0)] - [sL(y) - y(0)] - 2L(y) =$

$\Rightarrow (s^2 - s - 2)L(y) = sy(0) + y'(0) - y(0)$. $\Rightarrow L(y) = \dfrac{sy(0) + y'(0) - y(0)}{s^2 - s - 2}$ ▨

Remark 6.14: ① Let $f: [0,\infty) \to \mathbb{R}$ be a function. The Laplace trans $L(f)(s) = \int_0^{+\infty} e^{-st} f(t)\,dt$ can be defined for $s \in \mathbb{C}$.

② The inverse Laplace transform of a function F is given by

$(L^{-1}F)(t) = \frac{1}{2\pi i} \lim_{R \to +\infty} \int_{\gamma - iR}^{\gamma + iR} e^{st} F(s)\,ds$, where the integration is done along the vertical line $Re(s)=\gamma$ in the complex plane such that γ is greater than the real part of all singularities of F



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③ Let $f:[0,+\infty)\to\mathbb{R}$ be a continuous function of exponential order a. Then, $L^{-1}(L(f))(s)=f(s)$, for all $s\geq 0$. If $f:[0,+\infty)\to\mathbb{R}$ be a piecewise continuous function of exponential order a. Then, $L^{-1}(L(f))(s)=f(s)$, for almost every $s\geq 0$. $\boxed{///}$

We return to Example 6.13. We have $y=L^{-1}\left(\frac{sy(0)+y'(0)-y(0)}{s^2-s-2}\right)$. What is $L^{-1}\left(\frac{sy(0)+y'(0)-y(0)}{s^2-s-2}\right)$? We have $\frac{sy(0)+y'(0)-y(0)}{s^2-s-2}=\frac{y(0)}{s+1}+\frac{y'(0)+y(0)}{3}\left(\frac{1}{s-2}-\frac{1}{s+1}\right)$. By Example 6.5 and Thm 6.10, we see that $y(t)=y(0)e^{-t}+\frac{y'(0)+y(0)}{3}(e^{2t}-e^{-t})$.

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Example 6.15: By using Laplace transform to solve the nonhomogeneous O.D.E. $y''+by'+cy=f(t)$. $\Rightarrow [s^2L(y)-sy(0)-y'(0)]+b[sL(y)-y(0)]$
$+cL(y)=(Lf)(s)$. $\Rightarrow (s^2+bs+c)L(y)=(Lf)(s)+sy(0)+y'(0)+by(0)$.

$$\Rightarrow L(y)(s)=\frac{(s+b)y(0)+y'(0)}{s^2+bs+c}+\frac{(Lf)(s)}{s^2+bs+c}.$$

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Example 6.16: Find the solution of the O.D.E. $y''+y=\sin 2t$ with initial condition $y(0)=2$ and $y'(0)=1$. If y is the solution to the above O.D.E. and y, y' are of exponential order a. Then, $[s^2L(y)-sy(0)-y'(0)]+L(y)=L(\sin 2t)(s)$.

$\Rightarrow (s^2+1)L(y)-2s-1=L(\sin 2t)(s)$. Recall Example 6.7, $L(\sin 2t)(s)$ $=\frac{2}{s^2+4}$. $\Rightarrow L(y)=\frac{2s+1}{s^2+1}+\frac{2}{(s^2+4)(s^2+1)}=\frac{2s}{s^2+1}+\frac{5}{3}\frac{1}{s^2+1}-\frac{1}{3}\frac{2}{s^2+4}$.

By Example 6.7, $L(\sin t)=\frac{1}{s^2+1}$, $L(\sin 2t)=\frac{2}{s^2+4}$. $L(\cos t)(s)=$ $L(\sin' t)(s)=sL(\sin t)(s)-\sin 0=\frac{s}{s^2+1}$ $\Rightarrow L(2\cos t)(s)=\frac{2s}{s^2+1}$.

$\Rightarrow y(t)=2\cos t+\frac{5}{3}\sin t-\frac{1}{3}\sin 2t$.

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Example 6.17: Find the solution of the O.D.E. $y^{(4)}-y=0$

With initial condition $y(0)=y''(0)=y'''(0)=0$, $y'(0)=1$. By using the Laplace transform, we get $L(y^{(4)})-L(y)=0$. $L(y^{(4)})=s^4L(y)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)$. $\Rightarrow s^4L(y)-s^2-L(y)=0 \Rightarrow L(y)(s)=\frac{s^2}{s^4-1}=\frac{1}{4}\frac{1}{s-1}-\frac{1}{4}\frac{1}{s+1}+\frac{1}{2}\frac{1}{s^2+1}$. By Example 6.5, $L(e^t)(s)=\frac{1}{s-1}$, $L(e^{-t})(s)=\frac{1}{s+1}$. By Example 6.7, $L(\sin t)(s)=\frac{1}{s^2+1}$. $\Rightarrow y=\frac{1}{4}e^t-\frac{1}{4}e^{-t}+\frac{1}{2}\sin t$.

$\blacksquare$

Remark 6.18: Below are some useful formulas to solve O.D.E. by using Laplace transform. Let $a\in\mathbb{R}$. (i) $L(e^{at})=\frac{1}{s-a}$, (ii) $L(\sin at)=\frac{a}{s^2+a^2}$ (iii) $L(\cos at)=\frac{sa}{s^2+a^2}$.

$\blacksquare$

- Advantages of the Laplace transform method:

① Converting a problem of solving a differential equation to a problem of solving an algebraic equation.

② The dependence on the initial data is automatically build in.

③ Non-homogeneous equation can be treated in exactly the same way as the homogeneous equation.

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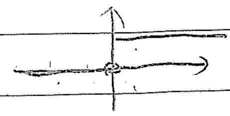
§6.3: Step function.

Def 6.19: The unit step function or Heaviside function is the function $H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$



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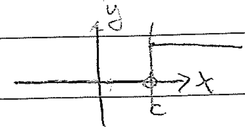
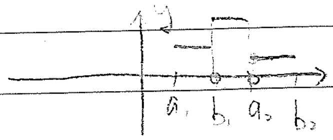
Example 6.20: ① For $c\in\mathbb{R}$, define $u_c(t) := H(t-c)$. The graph of u_c jump up from 0 to 1 at $t=c$.



② Let $a<b$. The characteristic function $1_{[a,b)}(t)$ is given by $1_{[a,b)}(t) = 1$ if $x\in[a,b)$, $1_{[a,b)}(x)=0$ if $x\notin[a,b)$.

Then, $1_{[a,b)}(t) = u_a(t) - u_b(t)$.

③ Let $a_1<b_1<a_2<b_2<\cdots<a_n<b_n$. The step function $f(t) = \sum_{j=1}^{n} \mu_j 1_{[a_j,b_j)}$ can be expressed by $f(t) = \sum_{j=1}^{n} \mu_j (u_{a_j}(t) - u_{b_j}(t))$, where $\mu_j\in\mathbb{R}$ $j=1,2,\ldots,n$.



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· The Laplace transform of u_c:

① If $c\leq 0$. Then, $\mathcal{L}(u_c)(s) = \int_0^\infty e^{-st} u_c(t)\,dt = \int_0^\infty e^{-st}\,dt = \frac{1}{s}$, for every $s>0$.

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③ If $c>0$, then $L(u_c)(s)=\int_0^{\infty} e^{-st}u_c(t)\,dt=\int_c^{+\infty} e^{-st}\,dt=$
$-\frac{1}{s}e^{-st}\Big|_c^{+\infty}=\frac{e^{-cs}}{s}$, for every $s>0$. $\Rightarrow L(u_c)(s)=\frac{e^{-\max\{c,0\}s}}{s}$.

Thm 6.3.1: Let $f:[0,+\infty)\to\mathbb{R}$ be a function such that the Laplace transform $L(f)(s)$ of f exists for $s>a\geq 0$. If c is a positive constant and $g(t):=u_c(t)f(t-c)$, then $L(g)(s)=e^{-cs}L(f)(s)$. Conversely if $G(s):=e^{-cs}L(f)(s)$, then $u_c(t)f(t-c)=(L^{-1}G)(t)$. ▨

pf $L(g)(s)=\lim_{R\to\infty}\int_c^R e^{-st}f(t-c)\,dt=\lim_{R\to\infty}\int_0^{R-c} e^{-s(t+c)}f(t)\,dt$
$=e^{-cs}\lim_{R\to\infty}\int_0^{R-c}e^{-st}f(t)\,dt=e^{-cs}L(f)(s)$. ▨

Example 6.22: Let $f:[0,+\infty)\to\mathbb{R}$ be defined by $f(t)=\begin{cases}\sin t, & 0\le t<\frac{\pi}{4}\\ \sin t+\cos(t-\frac{\pi}{4}), & t\ge\frac{\pi}{4}\end{cases}$.

Then, $f(t)=\sin t+u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4})$. $\Rightarrow L(f)(s)=L(\sin t)(s)+L(u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4}))(s)$. By Example 6.7, $L(\sin t)(s)=\frac{1}{s^2+1}$.

By Thm 6.21, $L(u_{\frac{\pi}{4}}(t)\cos(t-\frac{\pi}{4}))(s)=e^{-\frac{\pi}{4}s}L(\cos t)(s)=e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}$.

$\Rightarrow (Lf)(s)=\frac{1+se^{-\frac{\pi}{4}s}}{s^2+1}$. $\blacksquare$

Example 6.23: Find the inverse Laplace transform of $F(s)=\frac{1-e^{-2s}}{s^2}$. By Example 6.9, the inverse Laplace transform of s^{-2} is t. $(L^{-1}F)(t)=L^{-1}(s^{-2})(t)-L^{-1}(\frac{e^{-2s}}{s^2})=t-L^{-1}(\frac{e^{-2s}}{s^2})$. Let $G(s)=\underbrace{e^{-2s}}_{e^{-2s}}\underbrace{L(t)(s)}_{s^{-2}}$. By Thm 6.21, $(L^{-1}G)(t)=u_2(t)(t-2)$. $\Rightarrow (L^{-1}F)(t)=t-u_2(t)(t-2)$. $\blacksquare$

Thm 6.24: Let $f:[0,+\infty)\to\mathbb{R}$ be a function such that the laplace transform $L(f)(s)$ of f exists for all $s>a\ge 0$. If c is a constant and $g(t)=e^{ct}f(t)$, then $L(g)(s)=L(f)(s-c)$, for all $s>a+c$. Conversely, if $G(s)=L(f)(s-c)$, then $L^{-1}(G)(t)=e^{ct}f(t)$. $\blacksquare$

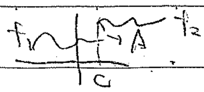
pf: $L(g)(s)=\int_0^{+\infty} e^{-st}e^{ct}f(t)dt=\int_0^{+\infty} e^{-(s-c)t}f(t)dt=L(f)(s-c).$ $\blacksquare$

Example 6.25: Find the inverse Laplace transform of $G(s)=\frac{1}{s^2-4s+5}$. $G(s)=\frac{1}{(s-2)^2+1}$. Let $f(t)=\sin t$ Then $L(f)(s)=\frac{1}{s^2+1}$.
$\Rightarrow G(s)=L(f)(s-2)$ By Thm 6.24, $L^{-1}(G)(t)=e^{2t}\sin t$. $\blacksquare$

Conclusion (important): Let $f:[0,+\infty)\to\mathbb{R}$ be a function such that the Laplace transform $L(f)(s)$ of f exists for all $s>a\geq 0$. ① Let $c>0$ and let $g(t)=u_c(t)f(t-c)$. Then, $L(g)(s)=e^{-cs}L(f)(s)$. Conversely, if $G(s)=e^{-cs}L(f)(s)$, then $L^{-1}(G)(t)=u_c(t)f(t-c)$. ② Let $c>0$ and let $g(t)=e^{ct}f(t)$. Then, $L(g)(s)=L(f)(s-c)$, for all $s>c+a$. Conversely, if $G(s)=L(f)(s-c)$, then $L^{-1}(G)(t)=e^{ct}f(t)$. $\blacksquare$

§ 6.4: Differential equations with discontinuous forcing functions.

Let $f:[0,+\infty)\to\mathbb{R}$ be a function defined by $f(t)=\begin{cases} f_1(t) & \text{if } 0\leq t<c,\\ f_2(t) & \text{if } t\geq c,\end{cases}$
where f_1, f_2 are continuous and $\lim_{t\to c^+} f_2(t)-\lim_{t\to c^-} f_1(t)=A$.

The point c is called a jump.  Define $g(t) := \begin{cases} f_1(t) & \text{if } 0 \le t < c \\ f_2(t) - Au_c(t) & \text{if } t \ge c. \end{cases}$

Then $g : [0, \infty) \to \mathbb{R}$ is continuous and $f = g + Au_c$. Similarly, if f is a piecewise continuous function which only has jump discontinuities $\{c_1, c_2, \cdots, c_n\}$ such that f is continuous on $[c_k, c_{k+1})$, for all $k = 1, 2, \cdots, n-1$. Let $c_0 = 0$, $c_{n+1} = +\infty$. We can write

$$f(t) = f(t)\mathbb{1}_{[c_0, c_1)}(t) + f(t)\mathbb{1}_{[c_1, c_2)}(t) + \cdots + f(t)\mathbb{1}_{[c_{n-1}, c_n)}(t) + f(t)\mathbb{1}_{[c_n, c_{n+1})}(t).$$

Let $A_k := \lim_{t \to c_k^+} f(t) - \lim_{t \to c_k^-} f(t)$, $k = 1, 2, \cdots, n$. Let $g : [0, \infty) \to \mathbb{R}$ be the function given by $g(t) = f(t) - A_1 u_{c_1}(t) - A_2 u_{c_2}(t) \cdots - A_n u_{c_n}(t)$

$= f(t) - \sum_{k=1}^{n} A_k u_{c_k}(t)$. Then g is continuous on $\mathbb{R}$ and $f = g + \sum_{k=1}^{n} A_k u_{c_k}$.

Consider the O.D.E. $y'' + by' + cy = f(t)$, where f is a piecewise continuous function which only has jump discontinuities $\{c_1, c_2, \cdots, c_n\}$ as described above.

Remark 6.25: The existence theorem cannot be applied due to the discontinuity of the forcing function. ////

Example 6.26 : Consider $y''+by'+cy=F1_{[\alpha,\beta)}(t)$, $y(0)=y_0$, $y'(0)=y_1$, where F is a constant, $0<\alpha<\beta$ and $c\neq 0$. Note that $1_{[\alpha,\beta)}(t) = u_\alpha(t)-u_\beta(t)$. By taking Laplace transform, we get $L(y'')(s) + bL(y')(s)+cL(y)(s) = L(F1_{[\alpha,\beta)}(t))(s)$. By Corollary 6.12,

$L(y'')(s) = s^2L(y)-sy(0)-y'(0)$, $L(y')(s)=sL(y)-y(0)$. $\Rightarrow$

$s^2L(y)(s)-sy(0)-y'(0)+b[sL(y)(s)-y(0)]+cL(y)(s) = L(F1_{[\alpha,\beta)}(t))(s)$.

Now, $L(F1_{[\alpha,\beta)}(t))(s) = FL(1_{[\alpha,\beta)}(t))(s) = FL(u_\alpha(t))(s)-FL(u_\beta(t))(s)$

Recall that $L(u_c(t))(s) = \frac{e^{-\max\{c,0\}s}}{s}$. Thus, $L(u_\alpha(t))(s) = \frac{e^{-\alpha s}}{s}$,

$L(u_\beta(t))(s) = \frac{e^{-\beta s}}{s}$. $\Rightarrow$ $s^2L(y)(s)-sy_0-y_1+b[sL(y)(s)-y_0] =$

$F\cdot\frac{e^{-\alpha s}-e^{-\beta s}}{s}$ $\Rightarrow$ $L(y)(s) = \frac{(s+b)y_0+y_1}{s^2+bs+c} + F\frac{e^{-\alpha s}-e^{-\beta s}}{s(s^2+bs+c)}$. What is y?

Consider homogeneous O.D.E. $z''+bz'+cz=0$, $z(0)=1$, $z'(0)=0$.

$\Rightarrow s^2L(z)(s)-sz(0)-z'(0)+b[sL(z)(s)-z(0)]+cL(z)(s)=0$

$\Rightarrow (s^2+bs+c)L(z)(s) = s+b$. $\Rightarrow L(z)(s) = \frac{s+b}{s^2+bs+c}$. Thus, $F\cdot\frac{e^{-\alpha s}-e^{-\beta s}}{s(s^2+bs+c)}$

$= F\cdot(e^{-\alpha s}-e^{-\beta s})\frac{1}{c}\left(\frac{1}{s}-\frac{s+b}{s^2+bs+c}\right) = F\cdot(e^{-\alpha s}-e^{-\beta s})\frac{1}{c}(L(1)(s)-L(z)(s))$

$= F\cdot(e^{-\alpha s}-e^{-\beta s})\cdot\frac{1}{c}L(1-z)(s)$. By Thm 6.21, $L^{-1}(F(e^{-\alpha s}-e^{-\beta s})\frac{1}{c}L(1-z)(s))$

$=\frac{F}{c}(u_\alpha(t)(1-z)(t-\alpha)-u_\beta(t)(1-z)(t-\beta))=\frac{F}{c}[u_\alpha(t)-u_\alpha(t)z(t-\alpha)$

$-u_\beta(t)+u_\beta(t)z(t-\beta)]$. Now consider the homogeneous O.D.E.

$Y''+bY'+cY=0,\ Y(0)=0,\ Y'(0)=1.\Rightarrow[s^2L(Y)(s)-sY(0)-Y'(0)]$

$+b[sL(Y)(s)-Y(0)]+cL(Y)(s)=0.\Rightarrow s^2L(Y)(s)-1+bsL(Y)(s)+cL(Y)(s)=0$

$\Rightarrow L(Y)(s)=\frac{1}{s^2+bs+c}.\Rightarrow\frac{(s+b)y_0+y_1}{s^2+bs+c}=y_0L(z)(s)+y_1L(Y)(s).\Rightarrow$

$y(t)=y_0z(t)+y_1Y(t)+\frac{F}{c}[u_\alpha(t)-u_\alpha(t)z(t-\alpha)-u_\beta(t)+u_\beta(t)z(t-\beta)]$.

· The first derivative of y: For $t\neq\alpha,\beta$, $y'(t)$ exists and we have

[※]

$y'(t)=y_0z'(t)+y_1Y'(t)+\frac{F}{c}[-u_\alpha(t)z'(t-\alpha)+u_\beta(t)z'(t-\beta)]$. Check the

differentiability of y at $t=\alpha$ and $t=\beta$ by looking at the limits

$\lim_{h\to0^-}\frac{y(c+h)-y(c)}{h}$ and $\lim_{h\to0^+}\frac{y(c+h)-y(c)}{h}$ for $c=\alpha,\beta$. We have $\lim_{h\to0^-}\frac{y(\alpha+h)-y(\alpha)}{h}$

$=y_0z'(\alpha)+y_1Y'(\alpha)+\frac{F}{c}\lim_{h\to0^-}\frac{u_\alpha(\alpha+h)(1-z(h))-u_\alpha(\alpha)(1-z(0))}{h}=y_0z'(\alpha)+y_1Y'(\alpha)$,

$\lim_{h\to0^+}\frac{y(c+h)-y(c)}{h}=y_0z'(\alpha)+y_1Y'(\alpha)+\frac{F}{c}\lim_{h\to0^+}\frac{u_\alpha(\alpha+h)(1-z(h))-u_\alpha(\alpha)(1-z(0))}{h}=y_0z'(\alpha)+$

$y_1Y'(\alpha)+\frac{F}{c}\lim_{h\to0^+}\frac{1-z(h)}{h}=y_0z'(\alpha)+y_1Y'(\alpha)-\frac{F}{c}z'(0)=y_0z'(\alpha)+y_1Y'(\alpha)\Rightarrow y'$ exists at $t=\alpha$

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and $y'(\alpha)=y_0z'(\alpha)+y_1\Upsilon'(\alpha)$. Therefore [*] holds for $t=\alpha$. Similarly, we can check that y' exists at β and $y'(\beta)=y_0z'(\beta)+y_1\Upsilon'(\beta)$ and [*] holds for $t=\beta$. Note that y' is continuous at α and β.

· The second derivative of y: For $t\neq\alpha,\beta$, we can check that

$$y''(t)=y_0z''(t)+y_1\Upsilon''(t)+\frac{F}{c}\left[u_\beta(t)z''(t-\beta)-u_\alpha(t)z''(t-\alpha)\right]. \quad [**]$$

$$\lim_{h\to 0^-}\frac{y'(\alpha+h)-y'(\alpha)}{h}=y_0z''(\alpha)+y_1\Upsilon''(\alpha)-\frac{F}{c}\lim_{h\to 0^-}\frac{u_\alpha(\alpha+h)z'(h)-u_\alpha(\alpha)z'(0)}{h}=y_0z''(\alpha)+y_1\Upsilon''(\alpha).$$

$$\lim_{h\to 0^+}\frac{y'(\alpha+h)-y'(\alpha)}{h}=y_0z''(\alpha)+y_1\Upsilon''(\alpha)-\frac{F}{c}\lim_{h\to 0^+}\frac{u_\alpha(\alpha+h)z'(h)-u_\alpha(\alpha)z'(0)}{h}$$

$$=y_0z''(\alpha)+y_1\Upsilon''(\alpha)-\frac{F}{c}\lim_{h\to 0^+}\frac{z'(h)-z'(0)}{h}=y_0z''(\alpha)+y_1\Upsilon''(\alpha)-\frac{F}{c}z''(0)$$

$$=y_0z''(\alpha)+y_1\Upsilon''(\alpha)+\frac{F}{c}\left[bz'(0)+cz(0)\right]=y_0z''(\alpha)+y_1\Upsilon''(\alpha)+F.$$

Since $F\neq 0$, we conclude that y'' at $t=\alpha$ does not exist. Similarly, y'' at $t=\beta$ does not exist. Now, for $t\neq\alpha,\beta$, we can check that

$$y''+by'+cy=y_0z''(t)+y_1\Upsilon''(t)+\frac{F}{c}\left[u_\beta(t)z''(t-\beta)-u_\alpha(t)z''(t-\alpha)\right]$$

$$+b(y_0z'(t)+y_1\Upsilon'(t))+\frac{bF}{c}\left[u_\beta(t)z'(t-\beta)-u_\alpha(t)z'(t-\alpha)\right]+c\left[y_0z(t)\right.$$

$$\left.+y_1\Upsilon(t)+\frac{F}{c}\left(u_\alpha(t)(1-z(t-\alpha))-u_\beta(t)(1-z(t-\beta))\right)\right]$$

$= \frac{F}{c}\left[u_\alpha(t)\left(bz'(t-\alpha)+cz(t-\alpha)\right)-u_\beta(t)\left(bz'(t-\beta)+cz(t-\beta)\right)\right]$

$+\frac{F}{c}\left[bu_\beta(t)\,z'(t-\beta)-bu_\alpha(t)\,z'(t-\alpha)\right]+\frac{F}{c}\left[cu_\alpha(t)(1-z(t-\alpha))-cu_\beta(t)\right.$

$\left.(1-z(t-\beta))\right] = F(u_\alpha(t)-u_\beta(t)) = F I_{[\alpha,\beta)}(t).$

$\blacksquare$

Def 6.27: Let $f: [0,+\infty)\to\mathbb{R}$ be a piecewise continuous function which has jump discontinuities $\{a,b\}$, $0<a<b$. A function y is said to be a solution to $y''+by'+cy=f(t)$ if y is continuously differentiable, y'' exists on $[0,a)\cup(a,b)\cup(b,+\infty)$, and $y''+by'+cy=f(t)$ on $[0,a)\cup(a,b)\cup(b,+\infty)$.

$\blacksquare$